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THE GENERAL HOMOGENIZED COMPOSITE SHELL MODEL AND THE APPLICATIONS

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Abstract—A regularly nonhomogeneous (composite), anisotropic, thin curved layer with rapidly oscillating material parameters and thickness is considered. It is assumed that the material parameters and thickness are periodic functions, and that thickness and period scale are both small and of the same order. The spatial elasticity problem for this composite layer is reduced to the new general homogenized shell model by means of the two-scale asymptotic homogenization method. The effective characteristics of a homogenized shell are expressed in terms of the solutions of the auxiliary local problems on the periodicity cell. The homogenized composite shell model is applied to the analysis of effective and local properties of some composite and reinforced shells. The effective characteristics of wafer-like, honeycomb-like and fibre-reinforced multilayer composite shells are described. The obtained formulae for the effective parameters refine the results of some known constructive-anisotropic theories.

INTRODUCTION

Today, the preponderance of uses for composite materials is in the form of shell and plate structural members whose strength and reliability, combined with reduced weight and concomitant material savings, offer the designer very impressive possibilities in some commercial applications. A fact which is of interest for us here is that whether the reinforcing effect comes from "real" elements (say, fibers) or "formal" ones (such as voids, holes and similar design features), it often happens that these form a regular structure with a period much smaller than the characteristic dimension of the structural member, with a consequence that the asymptotic homogenization analysis becomes applicable. The homogenized models of plates with periodical nonhomogeneities in tangential coordinate(s) have been developed in this way by Duvaut (1976) and some other workers (see Kalamkarov *et al.*, 1987a for a review). It should be noted, however, that the asymptotic homogenization method cannot be applied to a two-dimensional plate and shell theory if the space nonhomogeneities of the material vary on a scale comparable with the small thickness of the three-dimensional body under consideration. A refined approach developed by Caillerie (1984) consists of applying the two-scale formalism directly to the three-dimensional problem of a thin nonhomogeneous layer. An analogous approach was applied by Kohn and Vogelius (1984) in the problem of bending in thin homogeneous elastic plates with rapidly varying thickness.

In the present paper the modified homogenization method is applied to the study of a curved thin composite shell with a regular structure and wavy surfaces (Kalamkarov *et al.*, 1987b). The starting point is the exact three-dimensional formulation of the elasticity problem, without resource to the Kirchhoff–Love hypotheses or any similar simplifying assumptions. Owing to the presence of the small parameter δ, the original three-dimensional problem then proves to be amenable to a rigorous asymptotic analysis unifying an asymptotic three-to-two dimensions process and a homogenization composite material–homogeneous material process. The general homogenized model so obtained has many practical applications in the design of stiffened and composite thin-walled structures (see Parton *et al.*, 1991).

GENERAL HOMOGENIZED SHELL MODEL

Let us consider a periodically nonhomogeneous, thin curved layer with rapidly oscillating thickness and introduce orthogonal curvilinear nondimensional coordinates α_1, α_2, γ such that the coordinate lines α_1 and α_2 coincide with the main curvature lines of the midsurface of the shell and the γ axis is normal to the midsurface (Fig. 1). Let the cell of periodicity Ω_δ (Fig. 1) be determined by the following inequalities:

$$\left\{-\frac{\delta h_{1,2}}{2} < \alpha_1, \alpha_2 < \frac{\delta h_{1,2}}{2}, \gamma^- < \gamma < \gamma^+\right\}, \quad \gamma^\pm = \pm\frac{\delta}{2} \pm \delta F^\pm\left(\frac{\alpha_1}{\delta h_1}, \frac{\alpha_2}{\delta h_2}\right).$$

Here the small parameter δ is the thickness of the skin: parameters h_1 and h_2 have the orders of unity and determine the ratios of the tangential dimensions of the periodicity cell to the thickness. The periodic functions F^+ and F^- determine the shape of the upper and lower faces of the layer. For Lamé coefficients the following formulae are valid:

$$H_1 = A_1(1+k_1\gamma), \quad H_2 = A_2(1+k_2\gamma), \quad H_3 = 1,$$

where $A_1(\alpha_1, \alpha_2)$ and $A_2(\alpha_1, \alpha_2)$ are the coefficients of the first quadratic form and $k_1(\alpha_1, \alpha_2)$, $k_2(\alpha_1, \alpha_2)$ the main curvatures of the midsurface ($\gamma = 0$).

We begin by introducing the "rapid" coordinates of the problem,

$$y_1 = \alpha_1/(\delta h_1), \quad y_2 = \alpha_2/(\delta h_2), \quad z = \gamma/\delta,$$

in terms of which the unit cell Ω is defined by the inequalities

$$\{y_1, y_2 \in (-\tfrac{1}{2}, \tfrac{1}{2}), z \in (z^-, z^+)\}, \quad z^\pm = \pm\tfrac{1}{2} \pm F^\pm(y_1, y_2).$$

The regular nonhomogeneity of the material is mathematically modeled by the requirement that the stiffness tensor components $a_{ijmn}(y_1, y_2, z)$ be periodic with unit cell Ω in the coordinates y_1 and y_2, piecewise-smooth functions. They can have a finite number of discontinuities of the first kind on the nonintersecting contact surfaces between dissimilar constituents (such as matrix and fibers, binder and inclusions).

Following the asymptotic homogenization method (see Sanchez-Palencia, 1980; Kalamkarov *et al.*, 1987a, b) we postulate for the displacement vector components u_i the following two-scale expansion:

$$u_i = u_i^{(0)}(\alpha) + \delta u_i^{(1)}(\alpha, y_1, y_2, z) + \delta^2 u_i^{(2)}(\alpha, y_1, y_2, z) + \cdots,$$

where $\alpha = (\alpha_1, \alpha_2)$; $i = 1, 2, 3$; $u^{(l)}(\alpha, y_1, y_2, z)$ when $l = 1, 2, \ldots$ are periodic functions in y_1 and y_2 with periodicity cell Ω. It can be proved (see Kalamkarov *et al.*, 1987b; Parton *et al.*, 1991) that the following expressions for the main terms of the expansions for the component of the displacement vector u_i and stress tensor σ_{ij} are valid:

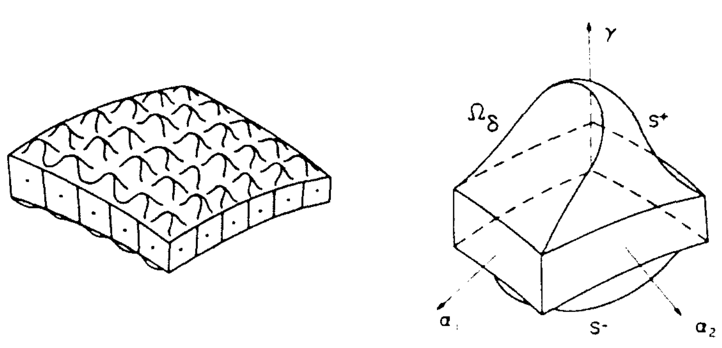

Fig. 1. Curvilinear thin regularly nonhomogeneous composite layer with wavy surfaces; unit cell Ω_δ.

$$\begin{cases} u_1 = v_1(x) - \delta \dfrac{z}{A_1} \dfrac{\partial w(x)}{\partial \alpha_1} + \delta U_1^{\mu\nu} \varepsilon_{\mu\nu} + \delta^2 V_1^{\mu\nu} \tau_{\mu\nu} + \cdots \\ u_2 = v_2(x) - \delta \dfrac{z}{A_2} \dfrac{\partial w(x)}{\partial \alpha_2} + \delta U_2^{\mu\nu} \varepsilon_{\mu\nu} + \delta^2 V_2^{\mu\nu} \tau_{\mu\nu} + \cdots \\ u_3 = w(x) + \delta U_3^{\mu\nu} \varepsilon_{\mu\nu} + \delta^2 V_3^{\mu\nu} \tau_{\mu\nu} + \cdots \end{cases} \quad (1)$$

$$\sigma_{ij} = b_{ij}^{\mu\nu} \varepsilon_{\mu\nu} + \delta c_{ij}^{\mu\nu} \tau_{\mu\nu} + \cdots . \quad (2)$$

These expressions determine the local structure of displacements and stresses with high accuracy. The summation convention applies whenever indices are repeated. Latin indices range from 1 to 3 and Greek ones from 1 to 2; $\varepsilon_{11} = \varepsilon_1$, $\varepsilon_{22} = \varepsilon_2$, $\varepsilon_{12} = \varepsilon_{21} = \omega/2$ are tension and shear deformations and $\tau_{11} = \aleph_1$, $\tau_{22} = \aleph_2$, $\tau_{12} = \tau_{21} = \tau$ are bending and torsion deformations of midsurface ($\gamma = \delta z = 0$). All these deformations can be expressed in terms of displacements of midsurface v_1, v_2 and w by well-known relations of the thin shell theory (see, for example, Novozhilov, 1962).

The coefficients in (2) are determined by the following expressions:

$$b_{ij}^{\mu\nu} = \frac{1}{h_\beta} a_{ijm\beta} \frac{\partial U_m^{\mu\nu}}{\partial \xi_\beta} + a_{ijm3} \frac{\partial U_m^{\mu\nu}}{\partial z} + a_{ij\mu\nu}$$

$$c_{ij}^{\mu\nu} = \frac{1}{h_\beta} a_{ijm\beta} \frac{\partial V_m^{\mu\nu}}{\partial \xi_\beta} + a_{ijm3} \frac{\partial V_m^{\mu\nu}}{\partial z} + z a_{ij\mu\nu}, \quad (3)$$

where $U_m^{\mu\nu}$ and $V_m^{\mu\nu}$ in (1) and (3) are functions of $\xi_1 = A_1 y_1$, $\xi_2 = A_2 y_2$ and z. These functions are periodic in variables ξ_1, ξ_2 with periods A_1 and A_2 respectively, and are determined by solutions of the following local problems:

$$\begin{cases} \dfrac{1}{h_\beta} \dfrac{\partial}{\partial \xi_\beta} b_{i\beta}^{\mu\nu} + \dfrac{\partial}{\partial z} b_{i3}^{\mu\nu} = 0 \\ \dfrac{1}{h_\beta} n_\beta^{\pm} b_{i\beta}^{\mu\nu} + n_3^{\pm} b_{i3}^{\mu\nu} = 0 \quad \text{at} \quad z = z^{\pm} \end{cases}$$

$$b_{ij}^{\mu\nu} \leftrightarrow c_{ij}^{\mu\nu}, \quad (4)$$

where $n_i^{\pm}$ are components of the normal to the upper (S^+) and lower (S^-) surfaces of the unit cell (Fig. 1) respectively, calculated in the coordinate system ξ_1, ξ_2 and z.

In the case of rigid contact on the surfaces of discontinuities of material parameters, the following conditions must be satisfied:

$$[\![U_m^{\mu\nu}]\!] = 0, \quad \left[\!\!\left[\frac{1}{h_\beta} n_\beta^{(c)} b_{i\beta}^{\mu\nu} + n_3^{(c)} b_{i3}^{\mu\nu} \right]\!\!\right] = 0$$

$$U_m^{\mu\nu} \leftrightarrow V_m^{\mu\nu}, \quad b_{ij}^{\mu\nu} \leftrightarrow c_{ij}^{\mu\nu} \quad (5)$$

where $n_i^{(c)}$ are components of the normal to the contact surface. The jump of a function f on the contact surface is denoted by $[\![f]\!]$.

It can be proved that the local problems (3)–(5) have unique solutions defined up to the additive constants (see Kalamkarov *et al.*, 1987b). These constants are determined by the following conditions:

$$\langle U_m^{\mu\nu} \rangle_\xi = 0 \quad \text{when} \quad z = 0, (U_m^{\mu\nu} \leftrightarrow V_m^{\mu\nu}). \quad (6)$$

Here the symbol $\langle \cdots \rangle_\xi$ means integration over the variables ξ_1 and ξ_2 only.

The elastic relations of the homogenized shell, that is, those between the stress and moment resultants on the one hand, and the midsurface deformations on the other, are found from (2). We have

$$
\left\{
\begin{aligned}
N_1 &= \delta\langle b_{11}^{\mu\nu}\rangle\varepsilon_{\mu\nu}+\delta^2\langle c_{11}^{\mu\nu}\rangle\tau_{\mu\nu}, \quad (1\leftrightarrow 2)\\
N_{12} &= \delta\langle b_{12}^{\mu\nu}\rangle\varepsilon_{\mu\nu}+\delta^2\langle c_{12}^{\mu\nu}\rangle\tau_{\mu\nu}\\
M_1 &= \delta^2\langle z b_{11}^{\mu\nu}\rangle\varepsilon_{\mu\nu}+\delta^3\langle z c_{11}^{\mu\nu}\rangle\tau_{\mu\nu}, \quad (1\leftrightarrow 2)\\
M_{12} &= \delta^2\langle z b_{12}^{\mu\nu}\rangle\varepsilon_{\mu\nu}+\delta^3\langle z c_{12}^{\mu\nu}\rangle\tau_{\mu\nu}.
\end{aligned}
\right. \tag{7}
$$

It can be proved that the following relations arise from (3)–(5):

$$
\begin{aligned}
&\langle b_{\alpha\beta}^{\mu\nu}\rangle = \langle b_{\mu\nu}^{\alpha\beta}\rangle, \quad \langle z b_{\alpha\beta}^{\mu\nu}\rangle = \langle c_{\mu\nu}^{\alpha\beta}\rangle,\\
&\langle z c_{\alpha\beta}^{\mu\nu}\rangle = \langle z c_{\mu\nu}^{\alpha\beta}\rangle, \quad (\alpha,\beta,\mu,\nu = 1,2).
\end{aligned} \tag{8}
$$

These relations provide the symmetry of the 6×6 coefficient matrix involved in the elastic constitutive equations (7), the matrix of effective elastic moduli. The averaging symbol $\langle f\rangle$ in (7) and (8) means the integration

$$\langle f\rangle = \int_{\Omega} f \,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}z.$$

The equations for a homogenized shell can be written in terms of stress and moment resultants in the following form:

$$
\left\{
\begin{aligned}
&\frac{\partial(A_2N_1)}{\partial\alpha_1}-\frac{\partial A_2}{\partial\alpha_1}N_2+\frac{\partial(A_1N_{12})}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}N_{12} = -A_1A_2G_1\\
&\frac{\partial(A_1N_2)}{\partial\alpha_2}-\frac{\partial A_1}{\partial\alpha_2}N_1+\frac{\partial(A_2N_{12})}{\partial\alpha_1}+\frac{\partial A_2}{\partial\alpha_1}N_{12} = -A_1A_2G_2\\
&k_1N_1+k_2N_2-\frac{1}{A_1A_2}\left[\frac{\partial(A_2Q_1)}{\partial\alpha_1}+\frac{\partial(A_1Q_2)}{\partial\alpha_2}\right] = G_3\\
&Q_1 = \frac{1}{A_1A_2}\left[\frac{\partial(A_2M_1)}{\partial\alpha_1}-\frac{\partial A_2}{\partial\alpha_1}M_2+\frac{\partial(A_1M_{12})}{\partial\alpha_2}+\frac{\partial A_1}{\partial\alpha_2}M_{12}\right]+m_1\\
&Q_2 = \frac{1}{A_1A_2}\left[\frac{\partial(A_1M_2)}{\partial\alpha_2}-\frac{\partial A_1}{\partial\alpha_2}M_1+\frac{\partial(A_2M_{12})}{\partial\alpha_1}+\frac{\partial A_2}{\partial\alpha_1}M_{12}\right]+m_2.
\end{aligned}
\right. \tag{9}
$$

In the above, the external loads are given by

$$
\begin{aligned}
G_i &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(\omega^+p_i^+ + \omega^-p_i^-)\,\mathrm{d}y_1\,\mathrm{d}y_2+\delta\langle P_i\rangle\\
m_\beta &= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}(\gamma^+\omega^+p_\beta^+ + \gamma^-\omega^-p_\beta^-)\,\mathrm{d}y_1\,\mathrm{d}y_2+\delta\langle \gamma P_\beta\rangle,
\end{aligned} \tag{10}
$$

where the functions $\omega^\pm$, defined by the formula

$$\omega^\pm = \left[1+\frac{1}{h_1^2}\left(\frac{\partial F^\pm}{\partial\xi_1}\right)^2+\frac{1}{h_2^2}\left(\frac{\partial F^\pm}{\partial\xi_2}\right)^2\right]^{1/2},$$

are determined by the profiles of the shell surfaces $\gamma = \gamma^\pm$; p_i^+ and p_i^- are the components of loads on the upper and lower faces of the layer, respectively, and P_i are the body force components, $i = 1, 2, 3$; $\beta = 1, 2$.

By substitution of relations (7) into eqns (9), we obtain the system of governing equations for v_1, v_2 and w. The boundary conditions can be given in the form accepted in the theory of thin elastic shells (see, for example, Novozhilov, 1962). In the simple case of homogeneous material and uniform thickness, all the local problems (3)–(6) can be solved exactly and the model obtained can be reduced to the known engineering formulation of thin anisotropic shell theory.

Let us consider now some applications of the general homogenized shell model.

THE ORTHOGONALLY STIFFENED SHELL

The periodicity cell of the orthogonally stiffened wafer-like shell is shown in Fig. 2. It consists from three elements, Ω_1, Ω_2 and Ω_3. The local problems have been solved in the case of small thickness of the cell elements and various anisotropic materials (see Kalamkarov, 1989). Formulae for effective stiffness moduli in the more simple case of isotropic material have the following form:

$$\langle b_{11}^{11}\rangle = D + E(F_2 + K_1), \quad \langle b_{22}^{22}\rangle = D + E(F_1 + K_1)$$

$$\langle b_{11}^{22}\rangle = \langle b_{22}^{11}\rangle = \nu D - EK_1, \quad \langle b_{12}^{12}\rangle = G$$

$$\langle c_{11}^{11}\rangle = E(S_2 + K_2), \quad \langle c_{22}^{22}\rangle = E(S_1 + K_2)$$

$$\langle c_{11}^{22}\rangle = \langle c_{22}^{11}\rangle = -EK_2, \quad \langle c_{12}^{12}\rangle = 0$$

$$\langle zc_{11}^{11}\rangle = D/12 + E(J_2 + K_3)$$

$$\langle zc_{22}^{22}\rangle = D/12 + E(J_1 + K_3)$$

$$\langle zc_{22}^{11}\rangle = \langle zc_{11}^{22}\rangle = \nu D/12 - EK_3$$

$$\langle zc_{12}^{12}\rangle = \frac{G}{12}\left\{1 + H^3\left(\frac{t_1}{h_1} + \frac{t_2}{h_2}\right) - \frac{96H^4}{\pi^5}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^5}\left[\frac{1}{A_1h_1}th\left(\frac{\pi n A_1 t_1}{2H}\right) + \frac{1}{A_2h_2}th\left(\frac{\pi n A_2 t_2}{2H}\right)\right]\right\}$$

$$D = \frac{E}{1-\nu^2}, \quad G = \frac{E}{2(1+\nu)}, \quad F_1 = \frac{t_1 H}{h_1}, \quad F_2 = \frac{t_2 H}{h_2}$$

$$S_1 = \frac{t_1(H^2+H)}{2h_1}, \quad J_1 = \frac{t_1(4H^3+6H^2+3H)}{12h_1}, \quad (1 \leftrightarrow 2). \tag{11}$$

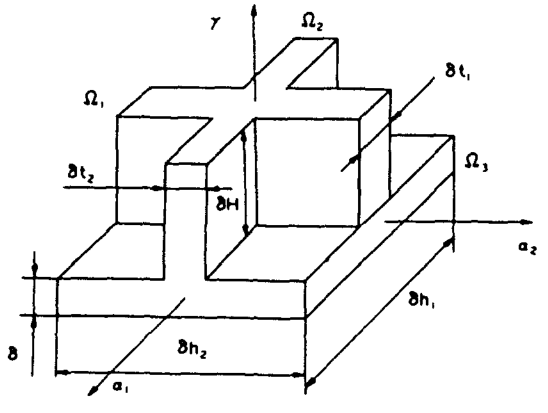

Fig. 2. Unit cell of a wafer shell in the coordinate system α_1, α_2, γ.

Here E is the Young's modulus; ν, is Poisson's ratio; the parameters K_1, K_2 and K_3 can be calculated from the obtained algebraic equations (see Parton *et al.*, 1989). The formulae (11) refine the effective parameters of stiffened shells in comparison with known constructive–anisotropic theory (according to this theory all of them are equal to zero). According to the calculations, the following conclusions can be drawn. (1) As found from formulae (11), the elastic moduli $\langle b_{11}^{11}\rangle$, $\langle b_{22}^{22}\rangle$, $\langle c_{11}^{11}\rangle$, $\langle c_{22}^{22}\rangle$, $\langle zc_{11}^{11}\rangle$ and $\langle zc_{22}^{22}\rangle$ can be calculated in the framework of the structurally anisotropic theory ($K_1 = K_2 = K_3 = 0$). (2) For the modulus $\langle b_{22}^{11}\rangle$, somewhat greater, but also reasonable, percentage changes are obtained (for example, 4% in the case $H = h_1 = h_2 = 10$, $t_1 = t_2 = 1$). (3) For the moduli $\langle c_{22}^{11}\rangle$ and $\langle zc_{22}^{11}\rangle$ more significant percentage changes occur. Note that while the change for $\langle zc_{22}^{11}\rangle$ is only of a quantitative nature, that for $\langle c_{22}^{11}\rangle$ is also interesting at a qualitative level because this modulus vanishes in view of (11) and is assumed to be zero in the framework of the structurally anisotropic theory of strengthened shells. One more point to be made is that the percentage changes increase with the height of the ribs (parameter H) and decrease with the distance between the ribs (parameters $h_{1,2}$). Using the solutions of the system (9), from (2) we are in a position to obtain the local stress distributions along the junctions of reinforcing elements. In particular, if we take $h_1 = h_2 = h$, then, in the case of simple bending ($\kappa_1 \neq 0$), the junctions of elements Ω_1 and Ω_2 will be subjected to the stresses

$$\sigma_{13}|_{x_1 = \pm\delta t/2,\, x_2 = 0} = \pm\delta\aleph_1(h/2)\varphi(z'), \quad (z' = z - \tfrac{1}{2}).$$

The function $\varphi(z')$ is shown graphically in Fig. 3. The curves marked 1 and 2 correspond, respectively, to the cases $H = 20$, $h_1 = h_2 = 60$, $t_1 = t_2 = t = 2$ and $H = h_1 = h_2 = 20$, $t_1 = t_2 = 0.5$.

THE HONEYCOMB-LIKE SHELL

The problem we consider here is that of a three-layered shell composed of a honeycomb filler of hexagonal structure sandwiched between two carrying layers as shown in Fig. 4. The calculation of the nonvanishing effective elastic moduli of the shell of Fig. 4 includes the solution of the local problems (3)–(6) and is somewhat lengthy to be reproduced here, so we only quote the final results of the calculation. For an isotropic material, both in the carrying layers (E_0, ν_0) and in the filler foil (E, ν), we have

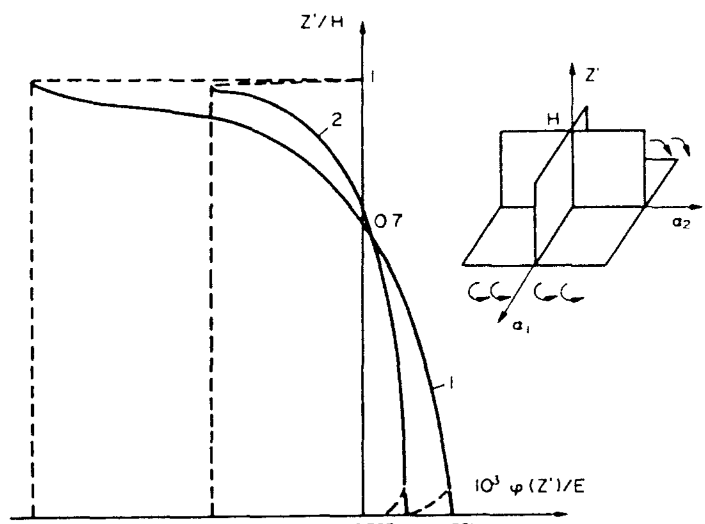

Fig. 3. Stress fields at the junctions of strengthening elements of a wafer shell under bending.

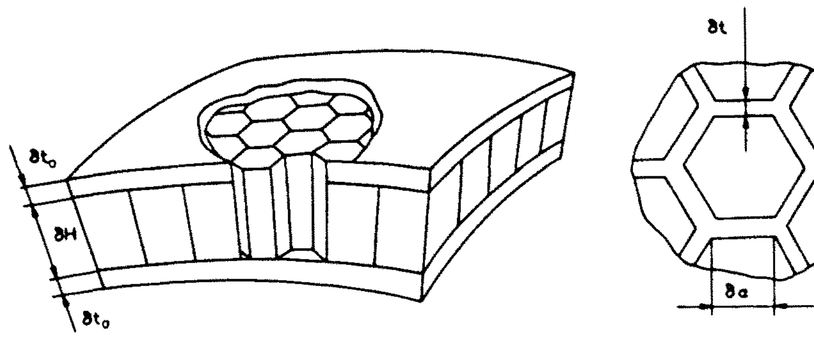

Fig. 4. Three-layered shell with a hexagonal honeycomb filler.

$$\langle b_{11}^{11}\rangle = \langle b_{22}^{22}\rangle = \frac{2E_0 t_0}{1-\nu^2} + \frac{\sqrt{3}}{4}\frac{EHt}{a}$$

$$\langle b_{11}^{22}\rangle = \langle b_{22}^{11}\rangle = \frac{2\nu_0 E_0 t_0}{1-\nu_0^2} + \frac{\sqrt{3}}{12}\frac{EHt}{a}$$

$$\langle b_{12}^{12}\rangle = \frac{E_0 t_0}{1+\nu_0} + \frac{\sqrt{3}}{12}\frac{EHt}{a}$$

$$\langle zc_{11}^{11}\rangle = \langle zc_{22}^{22}\rangle = \frac{E_0}{1-\nu^2}J_0 + \frac{\sqrt{3}}{48}\frac{EH^3 t}{a}$$

$$\langle zc_{11}^{22}\rangle = \langle zc_{22}^{11}\rangle = \frac{\nu_0 E_0}{1-\nu_0^2}J_0 + \frac{\sqrt{3}}{144}\frac{EH^3 t}{a}$$

$$\langle zc_{12}^{12}\rangle = \frac{E_0 J_0}{2(1+\nu_0)} + \frac{EH^3 t}{12\sqrt{3}(1+\nu)a}\left\{\frac{3+\nu}{4} - \frac{128H}{\pi^5 At}\right.$$
$$\left.\times \sum_{n=1}^{\infty} th[\pi(2n-1)At/(2H)](2n-1)^{-5}\right\}. \quad (12)$$

Here

$$A_1 = A_2 = A, \quad J_0 = H^2 t_0/2 + Ht_0^2 + 2t_0^3/3.$$

The first terms in (12) describe the contribution from the carrying layers, and the second terms describe that from the filler. It is seen that the contribution of the latter may be made comparable to, or even greater than, that of the former by appropriately varying the parameters E, H, t and a. The comparison of (12) with the results, obtained by different methods in the earlier work on the subject, shows that the greatest corrections occur in the elastic moduli $\langle b_{11}^{22}\rangle$ and $\langle zc_{11}^{22}\rangle$ (see Parton *et al.*, 1991).

THE MULTILAYER FIBRE-REINFORCED COMPOSITE SHELL

The term high-stiffness composite material usually refers to polymer matrix fiber reinforcement composites in which the Young's modulus of the fiber phase, E_f, is much larger than that of the matrix phase, E_m. Accordingly, the mechanical behavior of the composite will be predicted with an error of the order of only $E_m/E_f \ll 1$ if one assumes that, for comparable fiber and matrix percentage contents, the role of the matrix is negligible and the stressed state of the composite is determined by the deformation of the fiber system alone. The state of stress in the matrix itself will then be found from a problem set in the region occupied by the matrix, under appropriately formulated fiber–matrix interface conditions.

We thus consider a composite shell formed by N layers of parallel fibers, as shown in Fig. 5, and assume, in accordance with the above, that the fiber material is much stiffer than that of the matrix. In this case the solution of the local problems (3)–(6) is much simplified owing to the decoupling of the regions occupied by fibers and the matrix, and we will employ this fact when solving these problems for a fiber of the jth layer of the system (the fiber making an angle φ_j with the coordinate line α_1, the departure of the axis of a fiber from the shell midsurface, $\gamma = 0$, will be denoted by δa_j in coordinates α_1, α_2, γ). The exact solution of local problems was found for the case in which $a_j = 0$, under the assumption that the fibers were elliptic in cross-section (Kalamkarov, 1987). We use here this solution in order to get the following expressions for the effective elastic moduli of the composite shell:

$$\langle b_{\alpha\beta}^{\lambda\mu} \rangle = \sum_{j=1}^{N} E_j b_j \theta_j, \quad \langle c_{\alpha\beta}^{\lambda\mu} \rangle = \sum_{j=1}^{N} E_j b_j a_j \theta_j, \quad \langle z c_{\alpha\beta}^{\lambda\mu} \rangle = \sum_{j=1}^{N} E_j b_j \left[a_j^2 + \frac{1}{16}\left(1 + \frac{c_j}{1+v_j}\right)\right] \theta_j. \quad (13)$$

Here E_j and v_j are the material properties of fibers in the jth layer; θ_j is the volumetric fiber content in the jth layer; parameters b_j and c_j are determined from the following formulae for each combination of indices α, β, λ, $\mu = 1, 2$:

$$\alpha\beta\lambda\mu = 1111: \quad b_j = A_1^4 B_j^{-4} \cos^4 \varphi_j, \quad c_j = 2A_2^4 \tan^2 \varphi_j (1-e_j^2)\Delta_j;$$

$$\alpha\beta\lambda\mu = 2222: \quad b_j = A_2^4 B_j^{-4} \sin^4 \varphi_j, \quad c_j = 2A_1^4 c \tan^2 \varphi_j (1-e_j^2)\Delta_j;$$

$$\alpha\beta\lambda\mu = 1212: \quad b_j = A_1^2 A_2^2 B_j^{-4} \cos^2 \varphi_j \sin^2 \varphi_j,$$

$$c_j = \tfrac{1}{2}(A_1^4 \cot^2 \varphi_j + A_2^4 \tan^2 \varphi_j - 2A_1^2 A_2^2)(1-e_j^2)\Delta_j;$$

$$\alpha\beta\lambda\mu = 1122, 2211: \quad b_j = A_1^2 A_2^2 B_j^{-4} \cos^2 \varphi_j \sin^2 \varphi_j, \quad c_j = -2A_1^2 A_2^2 (1-e_j^2)\Delta_j;$$

$$\alpha\beta\lambda\mu = 1112, 1211: \quad b_j = A_1^3 A_2 B_j^{-4} \cos^3 \varphi_j \sin \varphi_j, \quad c_j = A_2^2 (A_2^2\, tg^2 \varphi_j - A_1^2)(1-e_j^2)\Delta_j;$$

$$\alpha\beta\lambda\mu = 1222, 2212: \quad b_j = A_1 A_2^3 B_j^{-4} \cos \varphi_j \sin^3 \varphi_j, \quad c_j = A_1^2 (A_1^2 c\, tg^2 \varphi_j - A_2^2)(1-e_j^2)\Delta_j. \quad (14)$$

The notation used in (14) is

$$B_j^2 = A_1^2 \cos^2 \varphi_j + A_2^2 \sin^2 \varphi_j, \quad \Delta_j = [B_j^2 + A_1^2 A_2^2 (1-e_j^2)]^{-1} \quad (15)$$

and e_j is the eccentricity of the elliptic cross-section of a fiber of the jth layer. Note that if we set $a_j = 0$ ($j = 1, 2, \ldots, N$) in expressions (13), these latter reduce to formulae for effective elastic moduli of a network-reinforced shell (see Kalamkarov, 1987).

It is of interest to compare the expressions (13)–(15) with similar results that have been derived from the structurally anisotropic model, the essential feature of which is that the average over the thickness of a multilayered shell is taken after first averaging the material characteristics of the constituent (orthotropic) layers. For the moduli $\langle b_{\alpha\beta}^{\lambda\mu} \rangle$ and $\langle c_{\alpha\beta}^{\lambda\mu} \rangle$, it is

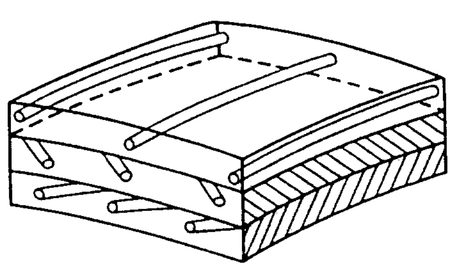

Fig. 5. Composite fiber-reinforced multilayer shell.

found that the expressions given by (13)–(15) are identical to the corresponding formulae for the generalized properties of a multilayered shell working in a tension–compression regime, provided the contribution of the matrix into the reduced properties of the orthotropic layers is negligible. The flexural and torsional stiffness moduli $\langle zc^{\lambda\mu}_{\alpha\beta}\rangle$ do differ from the corresponding results for the structurally anisotropic model and may be converted to these latter by setting $e_j = 1$ ($j = 1, 2, \ldots, N$) (which means a neglect of the shape of the cross-section) and replacing by 12 the factor 16 arising in the denominator through the moment of inertia of the elliptic fiber cross-section. The maximum percentage change in the values of the effective moduli is obtained for modulus $\langle zc^{22}_{11}\rangle$ (see Parton *et al.*, 1991).

CONCLUDING REMARKS

The proposed general model of a homogenized composite shell can be effectively applied to the analysis of highly heterogeneous shells and plates with regular structure (composite, porous, reinforced) with various stiffeners (wafer-like, rib-like, honeycomb-like, corrugated, network, etc.). The convergence of the solution of the three-dimensional elasticity problem for the curved layer to the solution and/or the homogenized shell model when the period and thickness tend to zero ($\delta \to 0$) can be proved by methods of the theory of homogenization under some additive assumptions concerning the functions determining the shape of the unit cell and the boundary of inclusions. The homogenized shell model makes it possible to calculate both the overall (effective) properties and local properties of various types of composite thin-walled structural members now widely used in many fields. It is not amiss to remark that the rigorous methods we present in the paper provide corrections, occasionaly appreciable ones, to effective moduli results that have been obtained earlier by other (approximate) methods.

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